proprep

Workbook

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Inner Product Spaces

Inner Product Spaces

Questions

1) For each two vectors $u=[x_1,x_2]$, $v=[y_1,y_2]$ in $\mathbb{R}^2$, we define:

$\langle u,v\rangle = x_1y_1-3x_1y_2-3x_2y_1+4x_2y_2$.

Check if this defines an inner product on $\mathbb{R}^2$.

2) For each two vectors $u=[x_1,x_2]$, $v=[y_1,y_2]$ in $\mathbb{R}^2$, we define:

$\langle u,v\rangle = x_1y_1-3x_1y_2-3x_2y_1+kx_2y_2$.

For which values of the parameter k does the above define an inner product on $\mathbb{R}^2$?

3) For each two vectors $u=[x_1,x_2,x_3]$, $v=[y_1,y_2,y_3]$ in $\mathbb{R}^3$, we define:

$<u,v>=x_1y_1+kx_1y_3+x_2y_2+kx_3y_1+x_3y_3$.

For which values of the parameter k does the above define an inner product on $\mathbb{R}^3$?

4) For each two vectors $u=[x_1,x_2,...,x_n]$, $v=[y_1,y_2,...,y_n]$ in $\mathbb{R}^n$, we define:

$<u,v>=\sum_{i=1}^{n}k_ix_iy_i$, where the parameters $k_1,...,k_n$ are positive numbers.

Show that the above definition gives an inner product on $\mathbb{R}^n$.
What do we get if $k_i=1$ for all $1\le i\le n$?

5) For each two matrices A, B in $M_{m\times n}[\mathbb{R}]$, we define: $\langle A,B\rangle = tr\left(B^TA\right)$.

Check if this defines an inner product on $M_{m\times n}[\mathbb{R}]$.

6) For each two functions f, g in $C[a,b]$, we define: $\langle f,g\rangle=\int_a^b f(x)\cdot g(x)dx$.

Check if this defines an inner product on $C[a,b]$.

Answer Key

1) Does not define.

2) $k>9$

3) $-1<k<1$

4) For the solution see the video.

5) For the solution see the video.

6) For the solution see the video.

Norm and Distance

Questions

1) Take the IPS $\mathbb{R}^3$, with the standard inner product*, and take the three vectors: $u=[1,-2,2]$, $v=[3,-2,6]$, $w=[5,3,-2]$, in $\mathbb{R}^3$. Compute the following:

a. $\langle u,v\rangle$ b. $\langle u,w\rangle$ c. $\langle v,w\rangle$ d. $\langle u+v,w\rangle$ e. $\|u\|$

f. $\|v\|$ g. $\|u+v\|$ h. $d(u,v)$ i. $\hat{u}$ j. $\hat{v}$

*AKA 'dot product' and we can write $u\cdot v$ instead of $\langle u,v\rangle$.

2) We are given three matrices $A=\begin{bmatrix}10 & 9 & 8\\ 7 & 6 & 5\end{bmatrix}$, $B=\begin{bmatrix}2 & 3 & 4\\ 5 & 6 & 7\end{bmatrix}$, $C=\begin{bmatrix}3 & -5 & 2\\ 1 & 0 & -4\end{bmatrix}$

in the IPS $M_{2\times 3}[\mathbb{R}]$, with inner product defined by $\langle X,Y\rangle = tr\left(Y^T X\right)$. Compute the following:

a. $\langle A,B\rangle$ b. $\langle A,C\rangle$ c. $\langle A,B+C\rangle$ d. $\langle B,C\rangle$

e. $\langle 4A+10B,11C\rangle$ f. $\|A\|$ g. $\|B\|$ h. $d(A,B)$ i. $\hat{A}$

3) We are given three functions $p(x)=x+3$, $q(x)=3x+1$, $r(x)=x^2-4x-1$

in the IPS $C[0,1]$, with inner product $\langle f,g\rangle=\int_0^1 f(x)\cdot g(x)dx$. Compute:

a. $\langle p,q\rangle$ b. $\langle p,r\rangle$ c. $\langle p,q+r\rangle$ d. $\|p\|$ e. $d(p,q)$ f. $\hat{r}$

4) Prove: $\|u+v\|^2=\|u\|^2+2\langle u,v\rangle+\|v\|^2$.

5) Prove: $\|u-v\|^2=\|u\|^2-2\langle u,v\rangle+\|v\|^2$.

6) Prove: $\langle u-v,u+v\rangle=\|u\|^2-\|v\|^2$.

7) Prove: $\|u+v\|^2+\|u-v\|^2=+2\|u\|^2+2\|v\|^2$. Give a geometric interpretation in the plane.

8) Prove: $\frac{1}{4}\left(\|u+v\|^2-\|u-v\|^2\right)=\langle u,v\rangle$.

Answer Key

1) a. 19 b. -5 c. -3 d. -8 e. 3 f. 7

g. $\sqrt{96}$ h. $\sqrt{20}$ i. $\left[\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right]$ j. $\left[\frac{3}{7},-\frac{2}{7},\frac{6}{7}\right]$

2) a. 185 b. -12 c. 173 d, -24 e. -3168 f. $\sqrt{355}$

g. $\sqrt{139}$ h. $\sqrt{124}$ i. $\hat{A}=\frac{1}{\sqrt{355}}\cdot\begin{bmatrix}10 & 9 & 8\\ 7 & 6 & 5\end{bmatrix}$

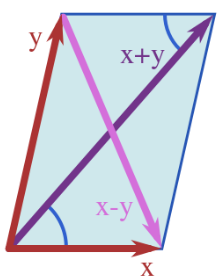
3) a. 9 b. -9.583333 c. -0.58333 d. $\sqrt{\frac{37}{3}}$ e. $\sqrt{\frac{4}{3}}$ f. $\hat{r}=\frac{r}{\|r\|}=\frac{x^2-4x-1}{\sqrt{7\frac{13}{15}}}$

4) For the solution see the video.

5) For the solution see the video.

6) For the solution see the video.

7) For the solution see the video. Geometric interpretation:



8) For the solution see the video.

Cauchy–Schwarz Inequality

Questions

1) Prove that if u,v are linearly dependent then $\left|\langle u,v\rangle\right| = \|u\|\cdot\|v\|$.

2) Let $x_1,x_2,\ldots,x_n$ and $y_1,y_2,\ldots,y_n$ be real numbers.

Prove that $\left(x_1y_1+x_2y_2+\ldots+x_ny_n\right)^2 \le \left(x_1^2+x_2^2+\ldots+x_n^2\right)\left(y_1^2+y_2^2+\ldots+y_n^2\right)$.

3) Let f,g be continuous functions on the closed interval $[a,b]$.

Prove that $\left(\int_a^b f(x)g(x)dx\right)^2 \le \left(\int_a^b f^2(x)\right)\left(\int_a^b g^2(x)\right)$.

4) Compute the angle between the vectors $u=[1,2,2]$, $v=[-2,1,2]$ in the IPS $\mathbb{R}^3$ with the standard inner product.

5) Compute the angle between the vectors $u=[3,4]$, $v=[1,2]$ in the IPS $\mathbb{R}^2$ with the inner product defined as follows: $\langle [x_1,x_2],[y_1,y_2]\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$.

6) Compute the angle θ between $p(x)=2x-1$ and $q(x)=x^2$ in the IPS $C[0,1]$ with inner product $\langle f,g\rangle = \int_0^1 f(x)g(x)dx$.

7) Compute the angle θ between $A=\begin{bmatrix}2 & 1\\ 3 & -1\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\ 2 & 3\end{bmatrix}$ in the IPS $M_2[\mathbb{R}]$ with inner product $\langle X,Y\rangle = tr(Y^TX)$.

Answer Key

1) Proof.

2) Proof.

3) Proof.

4) $\theta = 63.61^{\circ}$

5) $\theta = 9.44^{\circ}$

6) $\theta = 80^{\circ}$

7) $\theta = 89.97^{\circ}$

Orthogonality

Questions

1) Prove that the vectors $u=[1,2,3]$, $v=[4,7,-6]$ are orthogonal in $\mathbb{R}^3$.

2) Find the value of the parameter k, for which the vectors $u=[1,k,3]$, $v=[4,7,-6]$, are orthogonal in $\mathbb{R}^3$.

3) Find a unit vector perpendicular to the vectors $u=[1,2,3]$, $v=[2,5,7]$, in $\mathbb{R}^3$.

4) Show that the polynomials $p(x)=2x-1$, $q(x)=6x^2-6x+1$, are orthogonal in $C[0,1]$, with the inner product $\langle f,g\rangle=\int_0^1 f(x)\cdot g(x)dx$.

5) In the space $P_n[\mathbb{R}]$ (polynomials with degree $\le n$, over $\mathbb{R}$), we define an inner product as follows: $\langle p,q\rangle=\sum_{k=0}^{n} p(k)q(k)=p(0)q(0)+p(1)q(1)+...+p(n)q(n)$.
Show that the polynomials $p(x)=x(x-2)(x-4)(x-6)$, $q(x)=(x-1)(x-3)(x-5)(x-7)$ are orthogonal in $P_7[\mathbb{R}]$ with the inner product defined above.

6) Given two matrices $A=\begin{bmatrix} k & 1 \\ 3 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$, in $M_{2\times 2}[\mathbb{R}]$, with inner product $\langle X,Y\rangle=tr\left(Y^T X\right)$. For which value(s) of k are these matrices orthogonal?

7) Prove that $\|u+v\|=\|u-v\| \Leftrightarrow u\perp v$. Give a geometric interpretation in $\mathbb{R}^2$.

8) Prove that $\|u+v\|^2=\|u\|^2+\|v\|^2 \Leftrightarrow u\perp v$. Give a geometric interpretation in $\mathbb{R}^2$.

9) Prove that $\|u\|=\|v\| \Rightarrow (u-v)\perp(u+v)$. Give a geometric interpretation in $\mathbb{R}^2$.

Answer Key

1) Solution in the video.

2) $k = 2$

3) $\hat{w} = \frac{w}{\|w\|} = \frac{[-1,-1,1]}{\sqrt{1+1+1}} = \left[\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$

4) Solution in the video.

5) Solution in the video.

6) There are no such values.

7) Solution in the video.

8) Solution in the video.

9) Solution in the video.

Orthogonal Complement

Questions

1) Let $W = span\{[1,2,-1,1],[2,5,3,1]\}$, in $\mathbb{R}^4$ (standard inner product).

a. Find a basis and the dimension of $W^{\perp}$.
b. Show that the orthogonal decomposition theorem is satisfied.

2) Let $W = span\{[1,1,1]\}$ in $\mathbb{R}^3$.

a. Find a basis and the dimension of $W^{\perp}$.
b. Show that the orthogonal decomposition theorem is satisfied.

3) Consider the IPS $P_2[\mathbb{R}]$ with the inner product "borrowed" from $C[0,1]$: $\langle p,q\rangle = \int_0^1 p(x)\cdot q(x)dx$.

Let $W = span\{x\} \subseteq P_2[\mathbb{R}]$. Find a basis and the dimension of $W^{\perp}$.

4) Consider the IPS $P_2[\mathbb{R}]$ with the inner product "borrowed" from $C[0,1]$: $\langle p,q\rangle = \int_0^1 p(x)\cdot q(x)dx$.

Let $W = span\{x,x^2\} \subseteq P_2[\mathbb{R}]$. Find a basis and the dimension of $W^{\perp}$.

5) Let $W = span\left\{\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix},\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right\}$ in $M_{2\times 2}[\mathbb{R}]$, with inner product $\langle A,B\rangle = tr\left(B^T A\right)$.

Find a basis and the dimension of $W^{\perp}$.

6) Find the orthogonal complement, $W^{\perp}$, to the subspace W of 3×3 diagonal matrices in $M_3[\mathbb{R}]$ (with its usual inner product).

7) Find a basis for the orthogonal complement of the space, W, of symmetric 2×2 matrices, as a subspace of $M_2[\mathbb{R}]$ (with the usual inner product).

8) Suppose we're given a homogeneous $m\times n$ SLE: $A\cdot \underline{x} = \underline{0}$ (in matrix notation).
Let U be the solution space of the system in $\mathbb{R}^n$ (with usual inner=dot product).
Describe U using the concept of orthogonal complement and the concept of the row space of the matrix A.

9) Let W_1, W_2 be subsets in an IPS V. Prove that $W_1 \subseteq W_2 \Rightarrow W_2^{\perp} \subseteq W_1^{\perp}$.

10) Let W be a subspace of V (an inner product space). Prove that $W \subseteq W^{\perp\perp}$.

11) Let W be a subspace of V (an inner product space). Assume that V has finite dimension. Prove that $W = W^{\perp\perp}$.

12) Suppose that W_1, W_2 are subspaces of V (an IPS). Prove that $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

13) Suppose that W_1, W_2 are subspaces of V (an IPS). Prove that $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$.

Answer Key

1) a. $W^{\perp} = span\left\{\left[-3,1,0,1\right],\left[11,-5,1,0\right]\right\}$ and $\dim\left(W^{\perp}\right)=2$.

b. Solution in the recording.

2) a. $W^{\perp} = span\left\{\left[-1,0,1\right],\left[-1,1,0\right]\right\}$ and $\dim\left(W^{\perp}\right)=2$.

b. Solution in the recording.

3) $W^{\perp} = span\left\{-\frac{2}{3}+x,-\frac{1}{2}+x^{2}\right\}$ and $\dim\left(W^{\perp}\right)=2$.

4) $W^{\perp} = span\left\{3x^{2}-12x+10\right\}$ and $\dim\left(W^{\perp}\right)=1$.

5) $W^{\perp} = span\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right\}$ and $\dim\left(W^{\perp}\right)=2$..

6) $B_{W^{\perp}} = \left\{\begin{matrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\\ \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\end{matrix}\right\}$.

7) $B_{W^{\perp}} = \left\{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right\}$.

8) Solution in the video.

9) Solution in the video.

10) Solution in the video.

11) Solution in the video.

12) Solution in the video.

13) Solution in the video.

Orthogonal Sets and Bases

Questions

1) Given the set of vectors $S=\{[2,1,-4],[1,2,1],[3,-2,1]\}$ in $\mathbb{R}^3$.

a. Show that the set S is orthogonal.

b. Normalize vectors in S to obtain an orthonormal set.

c. Without computation, prove that S is a basis of $\mathbb{R}^3$.

2) Given the set of vectors $S=\{[2,1,-4],[1,2,1],[3,-2,1]\}$ in $\mathbb{R}^3$.

Write the vector $[13,-1,7]$ as a linear combination of the members of S,

by using inner products (and NOT row operations, echelon form, etc.).

3) Given the set of vectors $S=\{[2,1,-4],[1,2,1],[3,-2,1]\}$ in $\mathbb{R}^3$.

Write the coordinate vector of a general $[a,b,c]$ in $\mathbb{R}^3$ relative to S,

by using inner products and orthogonality.

Hint: We showed in an earlier exercise that S is an orthogonal basis of $\mathbb{R}^3$.

4) Suppose that $B=\{u_1,u_2,...,u_n\}$ is an orthogonal basis of V.

Prove that for all $v\in V$ $v=\frac{<v,u_1>}{<u_1,u_1>}u_1+\frac{<v,u_2>}{<u_2,u_2>}u_2+...+\frac{<v,u_n>}{<u_n,u_n>}u_n$.

Remark: the constants $a_i=\frac{<v,u_i>}{<u_i,u_i>}$ $(i=1,2,\ldots,n)$ are called the *Fourier coefficients*

[or *components*] of v relative to B.

5) Let $V=C[0,\pi]$ with the integral inner product and consider the set

$S=\{\cos x,\cos 2x,\cos 3x,...\}$ in V. Is S orthogonal? If so, is it orthonormal?

If it's orthogonal but not orthonormal, then normalize it.

6) Let $V=C[0,2\pi]$ with the integral inner product and consider the set

$S=\{1,\cos x,\sin x,\cos 2x,\sin 2x,...\}$ in V. Is S orthogonal? If so, is it orthonormal?

If it's orthogonal but not orthonormal, then normalize it.

7) Given the set $S=\{[2,4,4],[4,-1,-1],[0,2,-2]\}$ in $\mathbb{R}^3$ (usual inner product).

Is S an orthogonal set? If so, is it: a. orthonormal? b. a basis of $\mathbb{R}^3$?
If it's orthogonal but not orthonormal, then normalize it.

8) Given the set $S=\{1,x,x^2,x^3\}$ in $P_3[\mathbb{R}]$ with the integral inner product on $[0,1]$.

Is S an orthogonal set? If so, is it: a. orthonormal? b. a basis of $P_3[\mathbb{R}]$?

If it's orthogonal but not orthonormal, then normalize it.

9) Given $S=\{1,2x-1,6x^2-6x+1\}$ in $P_2[\mathbb{R}]$ with the integral inner product on $[0,1]$.

Is S an orthogonal set? If so, is it: a. orthonormal? b. a basis of $P_2[\mathbb{R}]$?

If it's orthogonal but not orthonormal, then normalize it.

10) Given a set $S=\left\{\begin{bmatrix}2&4&6\\0&2&4\\0&0&2\end{bmatrix},\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix},\begin{bmatrix}1&1&-1\\0&1&-1\\0&0&1\end{bmatrix}\right\}\subseteq M_3[\mathbb{R}]$ (standard inner product).

Is S an orthogonal set? If so, is it:
a. orthonormal?
b. a basis of $M_3[\mathbb{R}]$?

If it's orthogonal but not orthonormal, then normalize it.

Answer Key

1) Solution in the video.

2) $[13,-1,7] = -\frac{1}{7}[2,1,-4] + 3[1,2,1] + \frac{24}{7}[3,-2,1]$

3) $[a,b,c] = \frac{2a+b-4c}{21}[2,1,-4] + \frac{a+2b+c}{6}[1,2,1] + \frac{3a-2b+c}{14}[3,-2,1]$

4) Solution in the video.

5) It's orthonormal but not orthogonal; $\hat{S} = \left\{ \frac{\cos x}{\sqrt{0.5\pi}}, \frac{\cos 2x}{\sqrt{0.5\pi}}, \frac{\cos 3x}{\sqrt{0.5\pi}}, \ldots \right\}$.

6) It's orthogonal but not orthonormal; $\hat{S} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots \right\}$.

7) It's an orthogonal set, but (a.) not orthonormal, and (b.) it's a basis.

$$\hat{S} = \left\{ \frac{1}{\sqrt{36}}[2,4,4], \frac{1}{\sqrt{8}}[0,2,-2], \frac{1}{\sqrt{18}}[4,-1,-1] \right\}$$

8) It's not orthogonal.

9) It's orthogonal, but (a.) not orthonormal, and (b.) it's a basis.

$$\hat{S} = \left\{ 1, \sqrt{3}(2x-1), \sqrt{5}\left(6x^2-6x+1\right) \right\}$$

10) It's an orthogonal set, but (a.) not orthonormal and also (b.) not a basis.

$$\hat{S} = \left\{ \frac{1}{\sqrt{80}}\begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Gram Schmitt Process

Questions

1) Let $U = \text{span}\{[1,2,3],[4,5,6],[7,8,9]\} \subseteq \mathbb{R}^3$. Find an orthonormal basis for U.

2) Let $U = \text{span}\{[2,2,2,2],[1,1,2,4],[1,2,-4,-3]\} \subseteq \mathbb{R}^4$. Find an orthonormal basis for U.

3) Recall that $P_3(x)$ is the vector space of real polynomials of degree ≤ 3.

Note that $\{1, x, x^2, x^3\}$ is a basis for $P_3(x)$.

Find an orthonormal basis for $P_3(x)$ with the integral inner product* on the interval $[-1,1]$.

*Reminder: integral inner product on $[a,b]$ is defined by $\langle f, g \rangle = \int_a^b f(x)g(x)dx$.

4) Let U be the subspace of $M_2[\mathbb{R}]$ defined by $U = \text{span}\left\{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\right\}$.

Find an orthonormal basis of U with respect to the usual inner product* of $M_2[\mathbb{R}]$.

You may assume that $\left\{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\right\}$ is a linearly independent set.

*For the solutions see the videos.

Orthogonal Matrices

Questions

1) Determine which of the following matrices are orthogonal.

$$A=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}\quad B=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix}\quad C=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

For each orthogonal one, find its inverse.

2) a. Prove the theorem: If $A_{n\times n}$ is a square matrix, then A is orthogonal if and only if $A^T A = I$

b. Let $A_{n\times n}$ be an orthogonal matrix. Prove that A is invertible and $A^{-1} = A^T$.

3) [In this exercise, all matrices are $n\times n$].

a. Let A be an orthogonal matrix. Prove that A^T and A^{-1} are also orthogonal.

b. Let A, B be orthogonal matrices. Prove that AB is orthogonal.
Generalise this to the product of $k \geq 2$ orthogonal matrices.

c. Prove that the determinant of an orthogonal matrix is ± 1.

d. Is the sum of two orthogonal matrices always orthogonal?

e. Is a scalar multiple of an orthogonal matrix always orthogonal?

f. Show that a triangular orthogonal matrix is diagonal.

4) Let A be a square matrix of order n. True or false (prove or disprove):

a. The columns of A are an orthonormal basis of $\mathbb{R}^n$ if and only if its rows are.

b. The columns of A are an orthogonal basis of $\mathbb{R}^n$ if and only if its rows are.

5) a. Let *A* be a square matrix of order n, whose columns $\{v_1,...,v_n\}$ comprise an orthogonal basis of $\mathbb{R}^n$. Let $\lambda_i = v_i \cdot v_i$. Prove that $A^T A = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$.

b. Let *A* be as above. Prove that *A* can be inverted by performing these 2 steps:

1) Divide each column by the sum of the squares of its elements.
2) Transpose the matrix.

a. Use the above to find the inverse of $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

b. Use the above to find the inverse of $\begin{bmatrix} 1 & 1 & \sqrt{2} \\ 2 & 2 & -\sqrt{8} \\ -\sqrt{0.5} & \sqrt{0/5} & 0 \end{bmatrix}$.

6) Prove the following theorem:
Let *B* and *C* be two orthonormal bases of the vector space $\mathbb{R}^n$.
Then the change-of-basis matrix from *B* to *C* is an orthogonal matrix.

7) Let A be the change-of-basis matrix from an basis $B = \{u_1,...,u_n\}$ to a basis $C = \{v_1,...,v_n\}$ of $\mathbb{R}^n$ and assume A is orthogonal.

a. Prove that if B is an orthonormal basis then so is C.
b. Prove that if C is an orthonormal basis then so is B.

8) a. Let A be a real orthogonal matrix of order n. Prove that there exist orthonormal bases *B* and *C* of $\mathbb{R}^n$ such that A is the change-of-basis matrix from *B* to *C*.

b. Let $v \in \mathbb{R}^n$ be such that $\|v\| = 1$. Prove that there exists an orthogonal matrix who first column is the vector v.

Answer Key

1) A is orthogonal since $A^T A = I$.
B is orthogonal since $B^T B = I$.
C is not orthogonal.

2) Proof.

3) Proof.

4) a. True. b. False.

5) Proof.

6) Proof.

7) Proof.

8) Proof.

Orthogonal Transformations

Questions

1) Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove the following Theorem:
T is orthogonal if and only if $\|T(u)\| = \|u\| \ \ \forall u \in \mathbb{R}^n$.

2) Suppose that the linear transformation $T:\mathbb{R}^n \to \mathbb{R}^n$ is orthogonal. Prove that:

a. T is an isomorphism.

b. T^{-1} is orthogonal.

3) a. Let A be an orthogonal matrix of order n. Define a linear transformation $T:\mathbb{R}^n \to \mathbb{R}^n$ by $T(u) = Au$. Prove that T is an orthogonal transformation.

b. Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation. Prove that there exists an orthogonal matrix A such that $T(u) = Au, \forall u \in \mathbb{R}^n$.

4) a. Prove that the only possible eigenvalues of an orthogonal transformation are ± 1.
b. Prove that the only possible eigenvalues of an orthogonal matrix are ± 1.

5) Prove that the composition of orthogonal transformations on $\mathbb{R}^n$ is also orthogonal.
I.e. if $T_1, T_2, ..., T_k$ $(k \ge 2)$ are orthogonal then so is $T_1 \circ T_2 \circ ... \circ T_k$.

6) Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let $\{u_1, ..., u_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Prove that T is orthogonal if and only if $\{Tu_1, ..., Tu_n\}$ is orthonormal.
Notation: we can write $T(v)$ more briefly as Tv.

7) Let B be any orthonormal basis of $\mathbb{R}^n$. Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.
Let $[T]_B$ be the matrix representing T with respect to B.
Prove: T is an orthogonal transformation if and only if $[T]_B$ is an orthogonal matrix.

8) In each of the following, write the formula for the described T:

a. $T:\mathbb{R}^2 \to \mathbb{R}^2$ is the reflection transformation in the line $y = \dfrac{1}{\sqrt{3}}x$.

b. $T:\mathbb{R}^2 \to \mathbb{R}^2$ is the reflection transformation in the line $y = \sqrt{3}x$.

c. $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a rotation of 30°.

9) For each of the transformations defined below, verify that it's orthogonal and describe it in terms of a rotation or reflection.

a. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

c. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

d. $^{*}T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

* Solve with the help of the other parts

10) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a ccw rotation of angle θ around the origin.
Show that T is given by the formula $T[x,y] = [x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta]$.

11) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a reflection in the line that makes an angle of $\frac{\theta}{2}$ with the positive *x*–axis.
Show that T is given by the formula $T[x,y] = [x\cos\theta + y\sin\theta,\ x\sin\theta - y\cos\theta]$.

12) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.
Prove T is orthogonal if and only if T is a rotation or a reflection.

*Proof questions- for the solutions see the videos.

The Spectral Theorem

Questions

1) Let $A=\begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix}$ use the Spectral Theorem to diagonalize A, and to find an orthonormal basis of R^2.

2) Let $A=\begin{pmatrix} 3 & 2 & 4 \\ 2 & 5 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ use the Spectral Theorem to diagonalize A, and to find an orthonormal basis of R^3.

3) Let $A=\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ use the Spectral Theorem to diagonalize A, and to find an orthonormal basis of R^2.

4) Let $A=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}$ use the Spectral Theorem to diagonalize A, and to find an orthonormal basis of R^3.

5) Let $A=\begin{pmatrix} 4 & 3a-2 \\ a^2 & 4 \end{pmatrix}$, where $a \in R$.

a. Find the values of a or which A s symmetric.
b. Use the Spectral Theorem to diagonalize A.

6) Let $A=\begin{pmatrix} 8 & 5a-10 \\ a^2 & -2 \end{pmatrix}$, where $a \in R$. Is there a value of a for which A is symmetric. If it exists, Use the Spectral Theorem to diagonalize A.

7) Let $A=\begin{pmatrix} 9 & a \\ a & 9 \end{pmatrix}$, where $a \in R$. Use the Spectral Theorem to diagonalize A.

8) Let $A=\begin{pmatrix} 3 & \cos(\frac{\pi}{2}-a) & 0 \\ \sin a & 0 & \sin(2\pi-a) \\ 0 & \sin a & -6 \end{pmatrix}$ where $a \in R$.

a. Find the values of a or which A s symmetric.
b. Use the Spectral Theorem to diagonalize A **for one of those values**.

9) Prove or disprove

Let A be a real symmetric matrix with the following diagonalization: $A=P_1D_1P_1^t$ and B be a real symmetric matrix with the following diagonalization: $B=P_2D_2P_2^t$ where P1 and P2 are orthogonal matrices, so A+B can be diagonalized in the following form: $A+B=(P_1+P_2)D(P_1+P_2)^t$.

10) Prove or disprove

Let A be a real symmetric matrix, so there exists a matrix B, such that $A=B\bullet B^t$.

Answer Key

1) $\lambda_1 = 1, \lambda_2 = 7$.

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ and its columns are an orthonormal basis of R^2,

$D = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$, $P^t = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

2) $\lambda_1 = -1, \lambda_2 = 3, \lambda_3 = 9$;

$v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$P = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$ and its columns are an orthonormal basis of R^3。

$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$

$P^t = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$

3) $\lambda_1 = \lambda_2 = 3$.

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

P= $\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $P^t = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$

4) $\lambda_1 = 5, \lambda_2 = \lambda_3 = 3$. $v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \; P^t = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

5) a. $a = 1, 2$.

b. In case $a = 1$:

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}, \; \lambda_1 = 3, \lambda_2 = 5, \; v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; P = P^t = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \; D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}.$$

In case $a = 2$:

$$A = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}, \lambda_1 = 0, \lambda_2 = 8, \; v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, P = P^t = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}.$$

6) No Solution.

7) $\lambda_1 = -a+9, \lambda_2 = a+9$, $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $P = P^t = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, $D = \begin{pmatrix} -a+9 & 0 \\ 0 & a+9 \end{pmatrix}$.

8) a. $a = (2k+1)\bullet\pi, k \in \mathbb{Z}$.

b. For $a = \pi$, and in fact for every a that maintains these equations,

$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{pmatrix}$ and it's

very easy to diagonalize it.

9) This statement s false. If we take A and B such that:

$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $P_1 + P_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ isn't orthogonal.

10) This statement is true.