

# Workbook



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# Vector Spaces over $\mathbb{R}$ ( $\mathbb{R}^n$ )

## The Vector Space $\mathbb{R}^n$

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### Questions

- 1) Check if  $W$  is a subspace of  $\mathbb{R}^3$ , where  $W = \{\langle a, b, c \rangle \mid a + b + c = 0\}$ .
- 2) Check if  $W$  is a subspace of  $\mathbb{R}^3$ , where  $W = \{\langle a, b, c \rangle \mid a = c\}$ .
- 3) Check if  $W$  is a subspace of  $\mathbb{R}^3$ , where  $W = \{\langle a, b, c \rangle \mid a = 3b\}$ .
- 4) Check if  $W$  is a subspace of  $\mathbb{R}^3$ , where  $W = \{\langle a, b, c \rangle \mid a < b < c\}$ .
- 5) Check if  $W$  is a subspace of  $\mathbb{R}^3$ , where  $W = \{\langle a, b, c \rangle \mid a = c^2\}$ .
- 6) Check if  $W$  is a subspace of  $\mathbb{R}^3$ , where  $W = \{\langle a, b, c \rangle \mid c - b = b - a\}$ .
- 7) Check if  $W$  is a subspace of  $\mathbb{R}^3$ , where  $W = \{\langle a, b, c \rangle \mid b = a \cdot q, c = a \cdot q^2\}$ .

### Answer Key

- 1)  $W$  is a subspace of  $\mathbb{R}^3$ .
- 2)  $W$  is a subspace of  $\mathbb{R}^3$ .
- 3)  $W$  is a subspace of  $\mathbb{R}^3$ .
- 4)  $W$  is not a subspace of  $\mathbb{R}^3$ .
- 5)  $W$  is not a subspace of  $\mathbb{R}^3$ .
- 6)  $W$  is a subspace of  $\mathbb{R}^3$ .
- 7)  $W$  is not a subspace of  $\mathbb{R}^3$ .

## Vector Spaces over $\mathbb{R}$ ( $\mathbb{R}^n$ )

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### Linear Combination, Dependence and Span

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#### Questions

1) We are given the following vectors in  $\mathbb{R}^4$ :

$$u_1 = \langle 4, 1, 1, 5 \rangle, \quad u_2 = \langle 0, 11, -5, 3 \rangle, \quad u_3 = \langle 2, -5, 3, 1 \rangle, \quad u_4 = \langle 1, 3, -1, 2 \rangle.$$

- Is  $u_1$  a linear combination of  $u_4$ ?
- Does  $u_1$  belong to  $Sp\{u_4\}$ ?
- Is the set  $\{u_1, u_4\}$  linearly dependent?

2) We are given the following vectors in  $\mathbb{R}^4$ :

$$u_1 = \langle 4, 1, 1, 5 \rangle, \quad u_2 = \langle 0, 11, -5, 3 \rangle, \quad u_3 = \langle 2, -5, 3, 1 \rangle, \quad u_4 = \langle 1, 3, -1, 2 \rangle.$$

- Is  $u_3$  a linear combination of  $u_1$  and  $u_2$ ?
- Does  $u_3$  belong to  $Sp\{u_1, u_2\}$ ?
- Is the set  $\{u_1, u_2, u_3\}$  linearly dependent?  
If so, try to write each vector in the set as a linear combination of the others.

3) We are given the following vectors in  $\mathbb{R}^4$ :

$$u_1 = \langle 4, 1, 1, 5 \rangle, \quad u_2 = \langle 0, 11, -5, 3 \rangle, \quad u_3 = \langle 2, -5, 3, 1 \rangle, \quad u_4 = \langle 1, 3, -1, 2 \rangle.$$

- Is  $u_4$  a linear combination of  $u_1$  and  $u_2$ ?
- Does  $u_4$  belong to  $Sp\{u_1, u_2\}$ ?
- Is the set  $\{u_1, u_2, u_4\}$  linearly dependent?  
If so, try to write each vector in the set as a linear combination of the others.

4) We are given the following vectors in  $\mathbb{R}^4$ :

$$u_1 = \langle 4, 1, 1, 5 \rangle, \quad u_2 = \langle 0, 11, -5, 3 \rangle, \quad u_3 = \langle 2, -5, 3, 1 \rangle, \quad u_4 = \langle 1, 3, -1, 2 \rangle,$$

$$\text{and } v = \langle 4, 12, k, -2k \rangle.$$

- What should the value of  $k$  be, in order for the vector  $v$  to be a linear combination of  $u_1$  and  $u_2$ ?
- What should the value of  $k$  be, in order for the vector  $v$  to belong to  $Sp\{u_1, u_2\}$ ?
- What should the value of  $k$  be in order for the set  $\{u_1, u_2, v\}$  to be linearly dependent?

## Vector Spaces over $\mathbb{R}$ ( $\mathbb{R}^n$ )

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5) We are given the following vectors in  $\mathbb{R}^4$ :

$$u_1 = \langle 4, 1, 1, 5 \rangle, \quad u_2 = \langle 0, 11, -5, 3 \rangle, \quad u_3 = \langle 2, -5, 3, 1 \rangle, \quad u_4 = \langle 1, 3, -1, 2 \rangle,$$

$$\text{and } v = \langle a, b, c, d \rangle.$$

- a. What are the conditions on  $a, b, c, d$ , in order for  $v$  to be a linear combination of  $u_1$  and  $u_2$ ?
  - b. What are the conditions on  $a, b, c, d$ , in order for  $v$  to belong to  $Sp\{u_1, u_2\}$ ?
  - c. What are the conditions on  $a, b, c, d$ , in order for the set  $\{u_1, u_2, v\}$  to be linearly dependent?
- 6) Check if the following set of vectors is linearly dependent:  
 $A = \langle u = \langle 1, 2, 3 \rangle, v = \langle 4, 5, 6 \rangle, w = \langle 7, 8, 9 \rangle \rangle$
- 7) Check if the following set of vectors is linearly dependent:  
 $A = \langle u = \langle 1, 2, 3 \rangle, v = \langle 4, 5, 6 \rangle, w = \langle 7, 8, 10 \rangle \rangle$



## Vector Spaces over $\mathbb{R}$ ( $\mathbb{R}^n$ )

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### Basis for $\mathbb{R}^n$

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#### Questions

1) Check if each of the following sets is a basis of  $\mathbb{R}^3$  :

- $\{\langle 1, 0, 1 \rangle, \langle 0, 0, 1 \rangle\}$
- $\{\langle 1, 1, 2 \rangle, \langle 1, 2, 3 \rangle, \langle 3, 3, 4 \rangle, \langle 2, 2, 1 \rangle\}$
- $\{\langle 1, 2, 3 \rangle, \langle 4, 5, 6 \rangle, \langle 7, 8, 9 \rangle\}$

2) Check if each of the following sets is a basis of  $M_{2 \times 2}[\mathbb{R}]$  (AKA  $M_2[\mathbb{R}]$ ):

- $\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 9 & 1 \\ 2 & 3 \end{bmatrix} \right\}$
- $\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 9 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 16 \\ 7 & 8 \end{bmatrix} \right\}$
- $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

3) Check if each of the following sets is a basis of  $P_2[\mathbb{R}]$  (deg  $\leq 2$  poly):

- $\{1+x, x^2+2x+3\}$
- $\{1+x, x^2+2x+3, 2x+4x^2, x-x^2\}$
- $\{1+2x+3x^2, 4+5x+6x^2, 7+8x+10x^2\}$

4) Consider the following set of vectors in  $\mathbb{R}^3$  :  $T = \{\langle 1, 2, 3 \rangle, \langle 4, 5, 6 \rangle, \langle 7, 8, 9 \rangle, \langle 2, 3, 4 \rangle\}$ .

- Is  $T$  a basis of  $\mathbb{R}^3$  ?
- Find a maximal linearly independent subset  $T'$  of  $T$  .
- Extend  $T$  to a basis of  $\mathbb{R}^3$  .

### Answer Key

- 1) a. The two vectors can't form a basis for  $\mathbb{R}^3$ .  
b. The four vectors can't form a basis for  $\mathbb{R}^3$ .  
c. The vectors are linearly dependent and so are not a basis for  $\mathbb{R}^3$ .
- 2) a. The three vectors can't form a basis for  $M_{2 \times 2}[\mathbb{R}]$ .  
b. The five vectors can't form a basis for  $M_{2 \times 2}[\mathbb{R}]$ .  
c. The four vectors do form a basis for  $M_{2 \times 2}[\mathbb{R}]$ .
- 3) a. The two vectors can't form a basis for  $P_2[\mathbb{R}]$ .  
b. The four vectors can't form a basis for  $P_2[\mathbb{R}]$ .  
c. The three vectors do form a basis for  $P_2[\mathbb{R}]$ .
- 4) a. No, since it contains more than three vectors.  
b.  $T' = \{\langle 1, 2, 3 \rangle, \langle 4, 5, 6 \rangle\}$   
c.  $T^* = \{\langle 1, 2, 3 \rangle, \langle 4, 5, 6 \rangle, \langle 0, 0, 1 \rangle\}$

### Solution Space of Homogenous SLE

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#### Questions

1) Here are 3 systems of linear equations (SLEs):

$$1. \begin{cases} x + y - z + 2w = 0 \\ 3x - y + 7z + 4w = 0 \\ -5x + 3y - 15z - 6w = 0 \end{cases} \quad 2. \begin{cases} x - y + z + w = 0 \\ x + 2z - w = 0 \\ x + y + 3z - 3w = 0 \end{cases} \quad 3. \begin{cases} x - y + z + w = 0 \\ 2x - 2y + 2z + 2w = 0 \end{cases}$$

Let's denote by  $W$ ,  $U$  and  $V$  the subspaces spanned by 1., 2. and 3., respectively.

- Find a basis for each of  $W$ ,  $U$  and  $V$  as well as their dimension.
  - Find a basis for  $U + V$  and its dimension.
    - Find the dimension of  $U \cap V$ .
  - Find a basis for  $U \cap V$ .
- 2) Let  $U = \{ \langle a, b, c, d \rangle \in \mathbb{R}^4 \mid a = c, b = d \}$ . Find a basis and the dimension of  $U$ .
- 3) Let  $U = \{ \langle a, b, c, d \rangle \in \mathbb{R}^4 \mid c = a + b, d = b + c \}$ . Find a basis and the dimension of  $U$ .
- 4) Let  $U = \{ v \in \mathbb{R}^4 \mid v \cdot \langle 1, -1, 1, -1 \rangle = 0 \}$ . Find a basis and the dimension of  $U$ .

### Answer Key

- 1) a.  $B_W = \{\langle -1.5, 2.5, 1, 0 \rangle, \langle -1.5, -0.5, 0, 1 \rangle\}$   $\dim W = 2$ ,  
 $B_U = \{\langle -2, -1, 1, 0 \rangle, \langle 1, 2, 0, 1 \rangle\}$   $\dim U = 2$ ,  
 $B_V = \{\langle -1, 0, 0, 1 \rangle, \langle -1, 0, 1, 0 \rangle, \langle 1, 1, 0, 0 \rangle\}$   $\dim V = 3$
- b. i.  $B_{U+V} = \{\langle 0, 0, -1, 1 \rangle, \langle 0, 1, 1, 0 \rangle, \langle 1, 1, 0, 0 \rangle\}$   $\dim U + V = 3$
- b. ii.  $2 = \dim(U \cap V)$
- c.  $B_{U \cap V} = \{\langle -2, -1, 1, 0 \rangle, \langle 1, 2, 0, 1 \rangle\}$
- 2)  $B_U = \{\langle 0, 1, 0, 1 \rangle, \langle 1, 0, 1, 0 \rangle\}$ ,  $\dim U = 2$
- 3)  $B_U = \{\langle -1, 1, 0, 1 \rangle, \langle 2, -1, 1, 0 \rangle\}$ ,  $\dim U = 2$
- 4)  $B_U = \{\langle 1, 0, 0, 1 \rangle, \langle -1, 0, 1, 0 \rangle, \langle 1, 1, 0, 0 \rangle\}$   $\dim U = 3$

### Subspaces

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#### Questions

- 1) Consider the subspace of  $\mathbb{R}^4$  defined as follows:

$$U = \text{span}\{\langle 1, 1, -1, 2 \rangle, \langle 3, -1, 7, 4 \rangle, \langle -5, 3, -15, -6 \rangle\}.$$

- Find a basis for  $U$  and its dimension.
- Find a homogeneous SLE whose solution space is  $U$ .

- 2) Consider the subspace of  $\mathbb{R}^4$  defined as follows:

$$V = \text{span}\{\langle 1, 1, -1, 1 \rangle, \langle 1, 0, 2, -1 \rangle, \langle 1, 1, 3, -3 \rangle, \langle 5, 1, 5, 8 \rangle\}.$$

Find, for  $V$  : a basis, the dimension, a homogeneous SLE.

- 3) Consider the two subspaces of  $\mathbb{R}^4$  defined as follows:

$$U = \text{span}\{\langle 1, 1, -1, 2 \rangle, \langle 3, -1, 7, 4 \rangle, \langle -5, 3, -15, -6 \rangle\}$$

$$V = \text{span}\{\langle 1, -1, 1, 1 \rangle, \langle 1, 0, 2, -1 \rangle, \langle 1, 1, 3, -3 \rangle, \langle 5, 1, 5, 8 \rangle\}.$$

Find a basis and the dimension of  $U + V$ .

- 4) Consider the two subspaces of  $\mathbb{R}^4$  defined as follows:

$$U = \text{span}\{\langle 1, 1, -1, 2 \rangle, \langle 3, -1, 7, 4 \rangle, \langle -5, 3, -15, -6 \rangle\}$$

$$V = \text{span}\{\langle 1, -1, 1, 1 \rangle, \langle 1, 0, 2, -1 \rangle, \langle 1, 1, 3, -3 \rangle, \langle 5, 1, 5, 8 \rangle\}.$$

Find a basis and the dimension of  $U \cap V$ .

### Answer Key

- 1) a.  $B_U = \{\langle 1, 1, -1, 2 \rangle, \langle 0, -4, 10, -2 \rangle\}$ ,  $\dim U = 2$       b.  $\begin{cases} -3x + 5y + 2z = 0 \\ -3x - y + 2t = 0 \end{cases}$
- 2)  $-8x - y + 5z + 2t = 0$ ,  $B_V = \{\langle 1, -1, 1, 1 \rangle, \langle 0, 1, 1, -2 \rangle, \langle 0, 0, -2, 5 \rangle\}$ ,  $\dim V = 3$
- 3)  $B_{U+V} = \{\langle 1, 1, -1, 2 \rangle, \langle 0, -4, 10, -2 \rangle, \langle 0, 0, 1, 0 \rangle, \langle 0, 0, 0, 1 \rangle\}$ ,  $\dim U + V = 4$
- 4)  $\dim U \cap V = 1$ ,  $B_{U \cap V} = \{\langle 5, 1, 5, 8 \rangle\}$

### Row and Column Space

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#### Questions

- 1) Find a basis and the dimension of the row space and the column space of the following matrix. What is its rank?

$$\begin{bmatrix} 4 & 1 & 1 & 5 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 1 & 3 & -1 & 2 \end{bmatrix}$$

- 2) Find a basis and the dimension of the row space and the column space of the following matrix. What is its rank?

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 5 \\ 2 & 5 & 3 & 1 & 6 \\ 1 & -1 & -2 & 2 & 1 \\ -2 & 3 & 5 & -4 & -1 \end{bmatrix}$$

### Answer key

1)  $B_{row} = \{\langle 4, 1, 1, 5 \rangle, \langle 0, 11, -5, 3 \rangle\}$   $\dim(row) = 2$ .

$$B_{col} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -2 \\ 1 \end{bmatrix} \right\}, \dim(col) = 2; \text{rank} = 2.$$

2)  $B_{row} = \{\langle 1, 2, 1, 3, 5 \rangle, \langle 0, 1, 1, -5, -4 \rangle, \langle 0, 0, 0, 1, 1 \rangle\}$ ;  $\dim(row) = 3$ .

$$B_{col} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -16 \\ 37 \end{bmatrix} \right\}; \dim(col) = 3; \text{Rank} = 3.$$